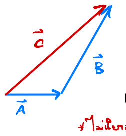
Vector Analysis

- Vector Algebra
- Coordinate systems
- Gradient, Divergence, Curl, Del Operator
- The Dirac Delta function
- Helmholtz's Theorem

▶ Vector Algebra

$\vec{A}=\vec{B}$: $|\vec{A}|=|\vec{B}|$ และมีทิศเดียวกัน ($\vec{A}$ และ $\vec{B}$) ไม่จำเป็นต้องเป็น vector เดียวกัน

Vector addition



$(\alpha+\beta)\vec{A} = \alpha\vec{A}+\beta\vec{A}$

$\alpha(\vec{A}+\vec{B}) = \alpha\vec{A}+\alpha\vec{A}$

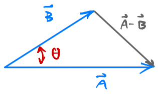
$\vec{A}+\vec{B} = \vec{B}+\vec{A}$

$(\vec{A}+\vec{B})+\vec{C} = \vec{A}+(\vec{B}+\vec{C})$

*ไม่นิยาม $\vec{A}+\alpha$ (vector + scalar)

Dot Product

$\boxed{\vec{A}\cdot\vec{B} = |\vec{A}||\vec{B}|\cos\theta}$ $0\le\theta\le180°$



$\vec{A}\cdot\vec{B} = \vec{B}\cdot\vec{A}$

$(\alpha\vec{A})\cdot\vec{B} = \vec{A}\cdot(\alpha\vec{B}) = \alpha(\vec{A}\cdot\vec{B})$

$\vec{A}\cdot(\vec{B}+\vec{C}) = \vec{A}\cdot\vec{B}+\vec{A}\cdot\vec{C}$

Cosine law : $C^2 = A^2+B^2-2AB\cos\theta$, $\vec{C}=\vec{A}-\vec{B}$

Cross Product

$\boxed{|\vec{A}\times\vec{B}| = |\vec{A}||\vec{B}|\sin\theta}$ $0\le\theta\le180°$

$\vec{A}\times\vec{A}=\vec{0}$

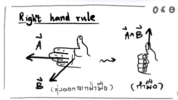
$\vec{A}\times\vec{B} = -\vec{B}\times\vec{A}$

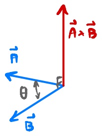
$\alpha(\vec{A}\times\vec{B}) = (\alpha\vec{A})\times\vec{B} = \vec{A}\times(\alpha\vec{B})$

$\vec{A}\times(\vec{B}+\vec{C}) = (\vec{A}\times\vec{B})+(\vec{A}\times\vec{C})$

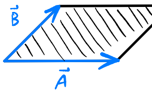
$\vec{A}\cdot(\vec{B}\times\vec{C}) = \vec{B}\cdot(\vec{A}\times\vec{C}) = \vec{C}\cdot(\vec{A}\times\vec{B})$ → triple products

$|\vec{A}\times\vec{B}|^2 = A^2B^2-(\vec{A}\cdot\vec{B})^2$ → Lagrange's identity



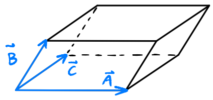


การใช้เวกเตอร์ในการหาพื้นที่



พื้นที่สี่เหลี่ยมด้านขนาน = $|\vec{A}\times\vec{B}|$

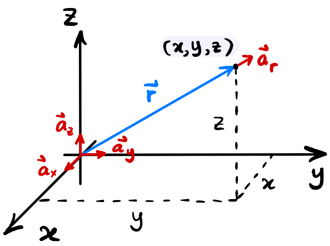
การใช้เวกเตอร์ในการหาปริมาตรทรงสี่เหลี่ยมด้านขนาน



ปริมาตรทรงสี่เหลี่ยมด้านขนาน = $|\vec{B}\cdot(\vec{A}\times\vec{C})|$

▶ Coordinate systems

Cartesian Coordinates
ระบบพิกัดฉาก



Position vector :

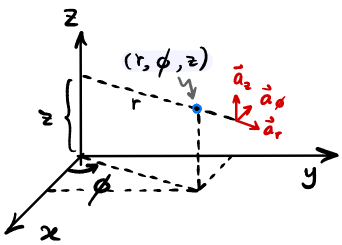
$\boxed{\vec{r} = x\vec{a}_x+y\vec{a}_y+z\vec{a}_z}$ ($\vec{a}_x=\hat{i}$, $\vec{a}_y=\hat{j}$, $\vec{a}_z=\hat{k}$)

$\vec{r} = r\vec{a}_r$

$r = \sqrt{x^2+y^2+z^2}$

$\vec{a}_r = \dfrac{\vec{r}}{r}$

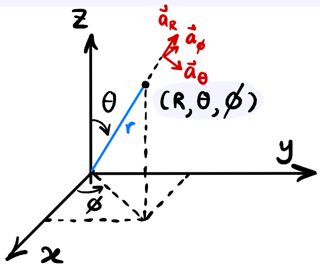
Cylindrical Coordinates
ระบบพิกัดทรงกระบอก



$x = r\cos\phi$	$r = \sqrt{x^2+y^2}$
$y = r\sin\phi$	$\phi = \arctan\left(\frac{y}{x}\right)$
$z = z$	$z = z$

$$\phi = \begin{cases} \arctan\frac{y}{x} & ; x>0 \\ \arctan\frac{y}{x}+\pi & ; x<0 \\ \pi/2 & ; x=0, y>0 \\ -\pi/2 & ; x=0, y<0 \end{cases}$$

Spherical Coordinates
ระบบพิกัดทรงกลม



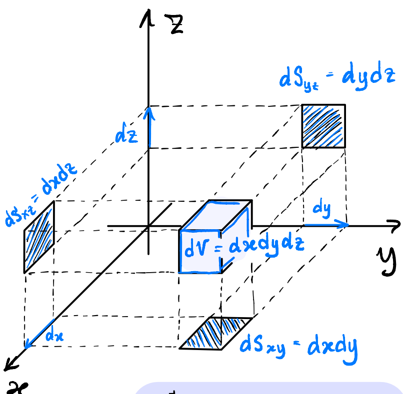
$x = R\sin\theta\cos\phi$	$R = \sqrt{x^2+y^2+z^2}$
$y = R\sin\theta\sin\phi$	$\theta = \arccos\left(\frac{z}{R}\right)$
$z = R\cos\theta$	$\phi = \arctan\left(\frac{y}{x}\right)$

Unit vector conversion :

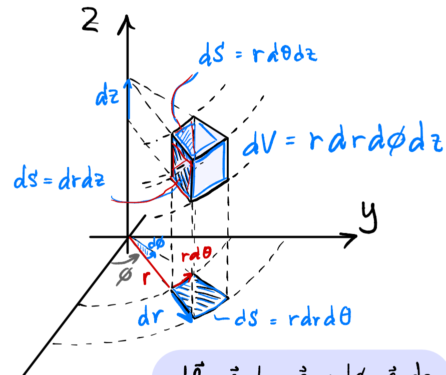
กระบอก	$\vec{a}_x = \vec{a}_r\cos\phi-\vec{a}_\phi\sin\phi$ $\vec{a}_y = \vec{a}_r\sin\phi+\vec{a}_\phi\cos\phi$ $\vec{a}_z = \vec{a}_z$	$\vec{a}_r = \vec{a}_x\cos\phi+\vec{a}_y\sin\phi$ $\vec{a}_\phi = -\vec{a}_x\sin\phi+\vec{a}_y\cos\phi$ $\vec{a}_z = \vec{a}_z$
ทรงกลม	$\vec{a}_x = \vec{a}_R\sin\theta\cos\phi+\vec{a}_\theta\cos\theta\cos\phi-\vec{a}_\phi\sin\phi$ $\vec{a}_y = \vec{a}_R\sin\theta\sin\phi+\vec{a}_\theta\cos\theta\sin\phi+\vec{a}_\phi\cos\phi$ $\vec{a}_z = \vec{a}_R\cos\theta-\vec{a}_\theta\sin\theta$	$\vec{a}_R = \vec{a}_x\sin\theta\cos\phi+\vec{a}_y\sin\theta\sin\phi+\vec{a}_z\cos\theta$ $\vec{a}_\theta = \vec{a}_x\cos\theta\cos\phi+\vec{a}_y\cos\theta\sin\phi-\vec{a}_z\sin\theta$ $\vec{a}_\phi = -\vec{a}_x\sin\phi+\vec{a}_y\cos\phi$

จาก \ กระบอก	$\vec{a}_x$	$\vec{a}_y$	$\vec{a}_z$
$\vec{a}_r$	$\cos\phi$	$\sin\phi$	0
$\vec{a}_\phi$	$-\sin\phi$	$\cos\phi$	0
$\vec{a}_z$	0	0	1

จาก \ ทรงกลม	$\vec{a}_x$	$\vec{a}_y$	$\vec{a}_z$
$\vec{a}_R$	$\sin\theta\cos\phi$	$\sin\theta\sin\phi$	$\cos\theta$
$\vec{a}_\theta$	$\cos\theta\cos\phi$	$\cos\theta\sin\phi$	$-\sin\theta$
$\vec{a}_\phi$	$-\sin\phi$	$\cos\phi$	0

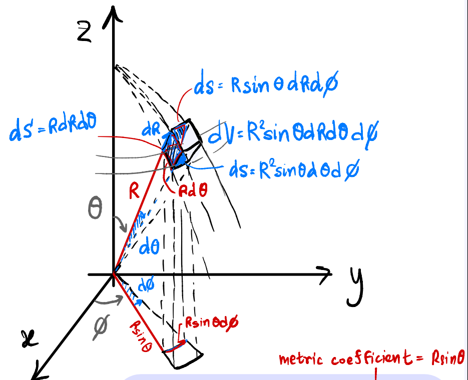


$d\vec{l} = \vec{a}_x dx+\vec{a}_y dy+\vec{a}_z dz$



$d\vec{l} = \vec{a}_r dr+\vec{a}_\phi r\,d\phi+\vec{a}_z dz$

metric coefficiet = r



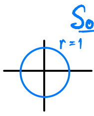
metric coefficient = $R\sin\theta$

$d\vec{l} = \vec{a}_R dR+\vec{a}_\theta R\,d\theta+\vec{a}_\phi R\sin\theta\,d\phi$

metric coefficient = R

(ex) Find the surface integral of $\vec{a}_r$ (cylindrical coordinates) when the surface is a unit circle.

Sol^n

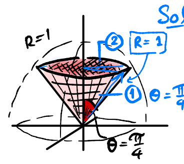


$$\iint_{\text{unit circle}} \vec{a}_r\, d\vec{s} = \int_0^{2\pi}\int_0^1 \vec{a}_r\, r\,dr\,d\phi$$

$$= \int_0^{2\pi}\int_0^1 (\vec{a}_x\cos\phi+\vec{a}_y\sin\phi)\, r\,dr\,d\phi$$

* (ex) Find the total interaction from the given surface.

Sol^n



$$\int_0^{2\pi}\int_0^{\frac{\pi}{4}} f(1,\theta,\phi)(1\,d\theta)(1\sin\theta\,d\phi) + \int_0^{2\pi}\int_0^1 f(R,\tfrac{\pi}{4},\phi)(dR)(R\sin\tfrac{\pi}{4}\,d\phi)$$

$$= \int_0^{2\pi}\int_0^{\pi/4} f(1,\theta,\phi)\cdot\sin\theta\,d\theta\,d\phi + \int_0^{2\pi}\int_0^1 f(R,\tfrac{\pi}{4},\phi)\,R\sin\tfrac{\pi}{4}\cdot dR\,d\phi$$

▶ Gradient of a Scalar Field

⇒ ∇T จะชี้ไปในทิศทางที่มีการเปลี่ยนแปลงสูงสุด และ $|\nabla T|$ จะบอกอัตราการเปลี่ยนแปลงในทิศทางนั้น

[Scalar → Vector]

$$\boxed{\nabla T = \text{grad}\, T = \vec{a}_{u_1}\frac{\partial T}{h_1\partial u_1} + \vec{a}_{u_2}\frac{\partial T}{h_2\partial u_2} + \vec{a}_{u_3}\frac{\partial T}{h_3\partial u_3}}$$

Del operator

$$\nabla \equiv \vec{a}_{u_1}\frac{\partial}{h_1\partial u_1} + \vec{a}_{u_2}\frac{\partial}{h_2\partial u_2} + \vec{a}_{u_3}\frac{\partial}{h_3\partial u_3}$$

└ metric coefficient ┘ ของแต่ละทิศทาง

$$\nabla T = \vec{a}_x\frac{\partial T}{\partial x} + \vec{a}_y\frac{\partial T}{\partial y} + \vec{a}_z\frac{\partial T}{\partial z} \quad \text{(Cartesian)}$$

$$\nabla T = \vec{a}_r\frac{\partial T}{\partial r} + \vec{a}_\phi\frac{1}{r}\frac{\partial T}{\partial \phi} + \vec{a}_z\frac{\partial T}{\partial z} \quad \text{(Cylindrical)}$$

$$\nabla T = \vec{a}_R\frac{\partial T}{\partial R} + \vec{a}_\theta\frac{1}{R}\frac{\partial T}{\partial R} + \vec{a}_\phi\frac{1}{R\sin\theta}\frac{\partial T}{\partial \phi} \quad \text{(Spherical)}$$

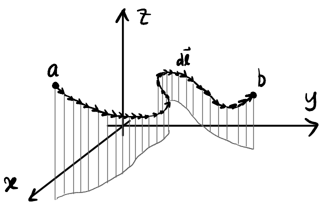
**note : สามารถหาเวกเตอร์ตั้งฉาก $\vec{N}$ ของผิว ϕ ได้จาก $\vec{N} = \nabla\phi$

(ex) Find the gradient of $r = \sqrt{x^2+y^2+z^2}$ (ขนาดของ position vector)

Sol^n $\nabla r = \vec{a}_x\frac{\partial r}{\partial x} + \vec{a}_y\frac{\partial r}{\partial y} + \vec{a}_z\frac{\partial r}{\partial z} = \frac{x\vec{a}_x + y\vec{a}_y + z\vec{a}_z}{\sqrt{x^2+y^2+z^2}} = \frac{\vec{r}}{r} = \vec{a}_r$ #

* หมายความว่าระยะทางจากจุดกำเนิดเพิ่มขึ้นเร็วที่สุดใน radial direction โดยมีอัตราการเพิ่มขึ้นเท่ากับ 1 *

▣ The Fundamental Theorem for Gradients



$$\boxed{\int_a^b (\nabla T)\cdot d\vec{l} = T(b) - T(a)}$$

↳ Corollaries :

① $\int_a^b (\nabla T)\cdot d\vec{l}$ ไม่ขึ้นกับเส้นทางจาก a ไป b

② $\oint (\nabla T)\cdot d\vec{l} = 0$ (วนกลับมาจุดเดิม)

(ex) Check the fundamental theorem for gradients, using $T = x^2 + 4xy + 2z^3$, the point $a = (0,0,0)$, $b = (1,1,1)$ and the following paths : (a) $(0,0,0)\to(1,0,0)\to(1,1,0)\to(1,1,1)$ (b) the parabolic path $z = x^2$; $y = x$

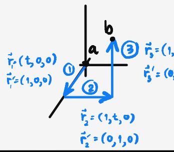
Sol^n $T(b) - T(a) = 7 - 0 = 7$

$\nabla T = (2x+4y)\vec{a}_x + (4x+2z^3)\vec{a}_y + (6yz^2)\vec{a}_z$

(a) $\int_a^b (\nabla T)\cdot d\vec{l} = \int_0^1 (\nabla T)\cdot(\vec{r}')\,dt$

$= \int_0^1 2t\,dt + \int_0^1 4\,dt + \int_0^1 6t^2\,dt$

$= 1 + 4 + 2 = 7$



(b) $\int_a^b (\nabla T)\cdot d\vec{l} = \int_0^1 (\nabla T)\cdot\vec{r}'\,dt$

$\vec{r} = (t, t, t^2)$, $\vec{r}' = (1, 1, 2t)$

$= \int_0^1 (2t+4t)\cdot dt + (4t+2t^6)\,dt + (6t^5)\cdot(2t)\,dt$

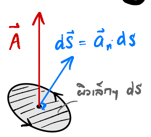
$= \int_0^1 14t^6 + 10t\,dt$

$= 2 + 5 = 7$

$\therefore \int_a^b (\nabla T)\cdot d\vec{l} = T(b) - T(a)$ ▣

▶ Divergence of a Vector Field

Divergence ใช้ในการบรรยายความสามารถในการถ่ายทอด (diverge) ของสนามออกจากจุดใดๆ ($\nabla\cdot\vec{A} = 0 \to$ Solenoidal field) } ลองจินตนาการถึงสระน้ำ → ก๊อกน้ำเป็นจุดที่ divergence เป็นบวก ↳ รูท่อน้ำทิ้ง เป็นจุดที่ divergence เป็นลบ



flux ของ $\vec{A}$ ที่ "พุ่งออก" จากปริมาตร δv

$$\boxed{\nabla\cdot\vec{A} = \text{div}\,\vec{A} \equiv \lim_{\delta v\to 0}\frac{\oint \vec{A}\cdot d\vec{s}}{\delta v}}$$

[vector → scalar]

$$\boxed{\nabla\cdot\vec{A} = \frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial u_1}(h_2h_3A_1) + \frac{\partial}{\partial u_2}(h_1h_3A_2) + \frac{\partial}{\partial u_3}(h_1h_2A_3)\right]}$$

$\vec{A} = A_1\vec{a}_{u_1} + A_2\vec{a}_{u_2} + A_3\vec{a}_{u_3}$

ทำการคูณแต่ละ component ด้วย metric coef. ของแกนอื่น แล้วจึงทำ partial dif จากนั้น ก็หารด้วย coef. ของทุกแกน

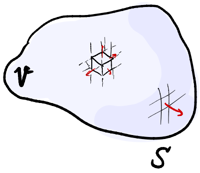
$$\nabla\cdot\vec{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} \quad \text{Cartesian}$$

$$\nabla\cdot\vec{A} = \frac{1}{r}\left[\frac{\partial(rA_r)}{\partial r} + \frac{\partial(A_\phi)}{\partial\phi} + \frac{\partial(rA_z)}{\partial z}\right] = \frac{1}{r}\frac{\partial(rA_r)}{\partial r} + \frac{1}{r}\frac{\partial(A_\phi)}{\partial\phi} + \frac{\partial A_z}{\partial z} \quad \text{Cylindrical}$$

$$\nabla\cdot\vec{A} = \frac{1}{R^2\sin\theta}\left[\frac{\partial(R^2\sin\theta\,A_R)}{\partial R} + \frac{\partial(R\sin\theta A_\theta)}{\partial\theta} + \frac{\partial(RA_\phi)}{\partial\phi}\right] = \frac{1}{R^2}\frac{\partial(R^2A_R)}{\partial R} + \frac{1}{R\sin\theta}\frac{\partial(\sin\theta A_\theta)}{\partial\theta} + \frac{1}{R\sin\theta}\frac{\partial(A_\phi)}{\partial\phi} \quad \text{Spherical}$$

▨ Divergence Theorem

/ Gauss' theorem

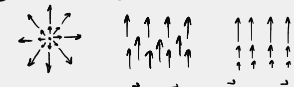


Total flux ของ $\vec{A}$ ที่พุ่งออกจากผิว S

$$\boxed{\iiint_V (\nabla\cdot\vec{A})\,dv = \oint_S \vec{A}\cdot d\vec{s}}$$

Volume integral ของ $\nabla\cdot\vec{A}$

(ex) จงหา divergence ของฟังก์ชันต่อไปนี้



$\vec{A}_a = x\vec{a}_x + y\vec{a}_y + z\vec{a}_z$, $\vec{A}_b = \vec{a}_z$, $\vec{A}_c = z\vec{a}_z$

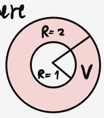
Sol^n $\nabla\cdot\vec{A}_a = \frac{\partial x}{\partial x} + \frac{\partial y}{\partial y} + \frac{\partial z}{\partial z} = 3$

$\nabla\cdot\vec{A}_b = \frac{\partial(1)}{\partial z} = 0$

$\nabla\cdot\vec{A}_c = \frac{\partial(z)}{\partial z} = 1$

(ex) Evaluate $\int_V \left(\nabla\cdot\frac{\cos^2\phi}{R^3}\vec{a}_R\right)dv$ & $\oint_S \frac{\cos^2\phi}{R^3}\vec{a}_R\cdot d\vec{s}$

Where V is the volume of the hollow sphere bounded by R=1 and R=3 and S is the surface enveloped V

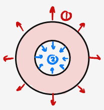


Sol^n : $D = \nabla\cdot\frac{\cos^2\phi}{R^3}\vec{a}_R = \frac{1}{R^2}\frac{\partial}{\partial R}\left(\frac{R^2\cos^2\phi}{R^3}\right) = -\frac{\cos^2\phi}{R^4}$ —(1)

$\int_V (D)\,dv = \int_0^{2\pi}\int_0^{\pi}\int_1^{3}\left(-\frac{\cos^2\phi}{R^4}\right)R^2\sin\theta\,dR\,d\theta\,d\phi$

$= \int_0^{2\pi}\cos^2\phi\,d\phi\int_0^{\pi}\sin\theta\,d\theta\int_1^{3}-\frac{1}{R^2}dR$

$= (\pi)(2)\left(-\frac{1}{2}\right) = \boxed{-\pi}$ Ans



$\oint_S\left(\frac{\cos^2\phi}{R^3}\vec{a}_R\right)\cdot d\vec{s} = \int_0^{2\pi}\int_0^{\pi}\left(\frac{\cos^2\phi}{3^3}\right)3^2\sin\theta\,d\theta\,d\phi + \int_0^{2\pi}\int_0^{\pi}\left(-\frac{\cos^2\phi}{1^3}\right)1^2\sin\theta\,d\theta\,d\phi$

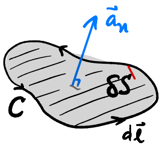
$= \frac{1}{3}\int_0^{2\pi}\cos^2\phi\,d\phi\int_0^{\pi}\sin\theta\,d\theta - \int_0^{2\pi}\cos^2\phi\,d\phi\int_0^{\pi}\sin\theta\,d\theta$

$= -\frac{2}{3}\int_0^{2\pi}\cos^2\phi\,d\phi\int_0^{\pi}\sin\theta\,d\theta$

$= -\frac{1}{2}(\pi)(2) = \boxed{-\pi}$ Ans

▶ Curl of a Vector Field

⇒ บรรยายการหมุนของสนามต่อพื้นที่หนึ่งๆ ($\nabla\times\vec{A} = 0 \to$ Irrotational field / Conservative field)



line integral ของ $\vec{A}$ บนวงปิด C ซึ่งวงปิด C ล้อมพื้นที่ δs ไว้

$$\boxed{\nabla\times\vec{A} \equiv \text{curl}\,\vec{A} \equiv \left(\lim_{\delta s\to 0}\frac{\oint \vec{A}\cdot d\vec{l}}{\delta s}\right)\vec{a}_n}$$

[vector → vector]

ทิศตั้งฉากกับผิว δs ตามหลักกฎมือขวา→

$$\boxed{\nabla\times\vec{A} = \frac{1}{h_1h_2h_3}\begin{vmatrix}\vec{a}_{u_1}h_1 & \vec{a}_{u_2}h_2 & \vec{a}_{u_3}h_3\\ \partial/\partial u_1 & \partial/\partial u_2 & \partial/\partial u_3\\ h_1A_1 & h_2A_2 & h_3A_3\end{vmatrix}}$$

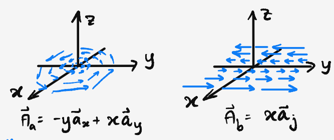
$\vec{A} = A_1\vec{a}_{u_1} + A_2\vec{a}_{u_2} + A_3\vec{a}_{u_3}$

$$\nabla\times\vec{A} = \begin{vmatrix}\vec{a}_x & \vec{a}_y & \vec{a}_z\\ \partial/\partial x & \partial/\partial y & \partial/\partial z\\ A_x & A_y & A_z\end{vmatrix} \quad \text{(Cartesian)}$$

$$\nabla\times\vec{A} = \frac{1}{r}\begin{vmatrix}\vec{a}_r & \vec{a}_\phi r & \vec{a}_z\\ \partial/\partial r & \partial/\partial\phi & \partial/\partial z\\ A_r & rA_\phi & A_z\end{vmatrix} \quad \text{(Cylindrical)}$$

$$\nabla\times\vec{A} = \frac{1}{R^2\sin\theta}\begin{vmatrix}\vec{a}_R & \vec{a}_\theta R & \vec{a}_\phi R\sin\theta\\ \partial/\partial R & \partial/\partial\theta & \partial/\partial\phi\\ A_R & RA_\theta & R\sin\theta A_\phi\end{vmatrix} \quad \text{(Spherical)}$$

(ex) จงหา curl ของฟังก์ชันต่อไปนี้



$\vec{A}_a = -y\vec{a}_x + x\vec{a}_y$, $\vec{A}_b = x\vec{a}_y$

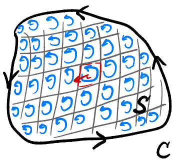
Sol^n $\nabla\times\vec{A}_a = \begin{vmatrix}\vec{a}_x & \vec{a}_y & \vec{a}_z\\ \partial/\partial x & \partial/\partial y & \partial/\partial z\\ -y & x & 0\end{vmatrix} = 2\vec{a}_z$, $\nabla\times\vec{A}_b = \begin{vmatrix}\vec{a}_x & \vec{a}_y & \vec{a}_z\\ \partial/\partial x & \partial/\partial y & \partial/\partial z\\ 0 & x & 0\end{vmatrix} = \vec{a}_z$

(ex) Find the curl of $\vec{A} = 5e^{-r}\cos\phi\,\vec{a}_r - 5\cos\phi\,\vec{a}_z$

Sol^n $\nabla\times\vec{A} = \frac{1}{r}\begin{vmatrix}\vec{a}_r & r\vec{a}_\phi & \vec{a}_z\\ \partial/\partial r & \partial/\partial\phi & \partial/\partial z\\ 5e^{-r}\cos\phi & 0 & -5\cos\phi\end{vmatrix}$

$= \frac{5\sin\phi}{r}\vec{a}_r + \frac{5e^{-r}\sin\phi}{r}\vec{a}_z$ ▣

▨ Stoke's Theorem



line integral ของ $\vec{A}$ บน C ที่ล้อม S

$$\boxed{\iint_S (\nabla\times\vec{A})\cdot d\vec{s} = \oint_C \vec{A}\cdot d\vec{l}}$$

Surface integral ของ $\nabla\times\vec{A}$

Corollaries :

① $\int_S (\nabla\times\vec{A})\,d\vec{s}$ ขึ้นกับเส้นขอบ ไม่ขึ้นกับรูปร่างของ S

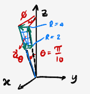
② $\oint_S (\nabla\times\vec{A})\,d\vec{s} = 0$ เพราะผิวปิดไม่มีเส้นขอบ

(ex) Given $\vec{A} = 6R\sin\phi\,\vec{a}_R + 18R\sin\theta\cos\phi\,\vec{a}_\phi$

Evaluate $\int_S(\nabla\times\vec{A})\cdot d\vec{s}$, $\oint_C \vec{A}\cdot d\vec{l}$. Where S is bounded by a cone $\theta = \frac{\pi}{10}$, R=2, R=4

$\phi = 0$, $\phi = \frac{3\pi}{10}$.

The direction of s is $+\vec{a}_\theta$



Sol^n $\nabla\times\vec{A} = \frac{1}{R\sin\theta}\begin{vmatrix}\vec{a}_R & R\vec{a}_\theta & R\sin\theta\,\vec{a}_\phi\\ \partial/\partial R & \partial/\partial\theta & \partial/\partial\phi\\ 6R\sin\phi & 0 & (R\sin\theta)18R\sin\theta\cos\phi\end{vmatrix}$

$= (36\cos\theta\cos\phi)\vec{a}_R + \frac{1}{\sin\theta}(6\cos\phi - 36\sin^2\theta\cos\phi)\vec{a}_\theta$

$\int_S(\nabla\times\vec{A})\cdot d\vec{s} = \int_0^{\frac{3\pi}{10}}\int_2^4 (\nabla\times\vec{A})\cdot R\sin\theta\,dR\,d\phi\,\vec{a}_\theta$

$= \int_0^{\frac{3\pi}{10}}\int_2^4 (6\cos\phi - 36\sin^2\theta\cos\phi)\cdot R\,dR\,d\phi$; $\theta = \frac{\pi}{10}$

$= \boxed{12.44}$

$\oint_C \vec{A}\cdot d\vec{l} = \int_0^{\frac{3\pi}{10}} 18R\sin\theta\cos\phi\cdot R\sin\theta\,d\phi$ $(R=4, \theta=\frac{\pi}{10})$

$+ \int_4^2 6R\sin\phi\,dR$ $(\phi = \frac{3\pi}{10})$

$+ \int_{\frac{3\pi}{10}}^{0} 18R\sin\theta\cos\phi\cdot R\sin\theta\,d\phi$ $(R=2, \theta=\frac{\pi}{10})$

$+ \int_2^4 6R\sin\phi\,dR$ $(\phi = 0)$

$= \boxed{12.44}$

Del Operator Identities

(1) $\nabla(fg) = f(\nabla g) + g(\nabla f)$

(2) $\nabla(\vec{A}\cdot\vec{B}) = \vec{A}\times(\nabla\times\vec{B}) + \vec{B}\times(\nabla\times\vec{A}) + (\vec{A}\cdot\nabla)\vec{B} + (\vec{B}\cdot\nabla)\vec{A}$

(3) $\nabla\cdot(f\vec{A}) = f(\nabla\cdot\vec{A}) + \vec{A}\cdot(\nabla f)$

(4) $\nabla\cdot(\vec{A}\times\vec{B}) = \vec{B}\cdot(\nabla\times\vec{A}) - \vec{A}\cdot(\nabla\times\vec{B})$

(5) $\nabla\times(f\vec{A}) = f(\nabla\times\vec{A}) - \vec{A}\times(\nabla f)$

(6) $\nabla\times(\vec{A}\times\vec{B}) = (\vec{B}\cdot\nabla)\vec{A} - (\vec{A}\cdot\nabla)\vec{B} + \vec{A}(\nabla\cdot\vec{B}) - \vec{B}(\nabla\cdot\vec{A})$

(7) $\nabla\times(\nabla f) = 0$: Curl-free, irrotational, conservative field

(8) $\nabla\cdot(\nabla\times\vec{A}) = 0$: Divergenceless, solenoidal field

Two Null Identities

(9) $\nabla\cdot(\nabla T) = \nabla^2 T$ Laplacian

(10) $\nabla\times(\nabla\times\vec{A}) = \nabla(\nabla\cdot\vec{A}) - \nabla^2\vec{A}$

Laplacian

Scalar : $\nabla^2 T = \nabla\cdot(\nabla T)$

Vector : $\nabla^2\vec{A} = \nabla(\nabla\cdot\vec{A}) - \nabla\times(\nabla\times\vec{A}) = \frac{\partial^2\vec{A}}{\partial x^2} + \frac{\partial^2\vec{A}}{\partial y^2} + \frac{\partial^2\vec{A}}{\partial z^2}$ (Cartesian)

Scalar

$$\nabla^2 T = \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} \quad \text{(Cartesian)}$$

$$\nabla^2 T = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 t}{\partial \phi^2} + \frac{\partial^2 t}{\partial z^2} \quad \text{(Cylindrical)}$$

$$\nabla^2 T = \frac{1}{R^2}\frac{\partial}{\partial R}\left(R^2\frac{\partial T}{\partial T}\right) + \frac{1}{R^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial T}{\partial\theta}\right) + \frac{1}{R^2\sin^2\theta}\frac{\partial^2 T}{\partial\phi^2} \quad \text{(Spherical)}$$

ex Calculate the Laplacian of the following functions.

(a) $T_a = x^2 + 2xy + 3z + 4 \longrightarrow \nabla^2 T_a = 2 + 0 + 0 = \underline{2}$

(b) $T_b = \sin x \sin y \sin z \longrightarrow \nabla^2 T_b = \underline{-3\sin x \sin y \sin z}$

(c) $T_c = e^{-5x}\sin 4y \sin 3z \longrightarrow \nabla^2 T_c = 25e^{-5x}\sin 4y\sin 3z - 16e^{-5x}\sin 4y\sin 3z - 9e^{-5x}\sin 4y \sin 3z = \underline{0}$

(d) $\vec{D} = x^2\vec{a}_x + 3xz^2\vec{a}_y - 2xz\vec{a}_z \rightarrow (2\vec{a}_x) + (\vec{0}) + (6xz\vec{a}_y) = \underline{2\vec{a}_x + 6xz\vec{a}_y}$

The Dirac Delta Function

The one-dimensional Dirac Delta Function	The three-dimensional Dirac Delta Function
$\delta(x) = \begin{cases} 0, & x \neq 0 \\ \infty, & x = 0 \end{cases}$ และ $\int_{-\infty}^{\infty}\delta(x)\,dx = 1$ $\int_{-\infty}^{\infty} f(x)\delta(x-a)\,dx = f(a)$	$\delta^3(\vec{r}) = \delta(x)\delta(y)\delta(z)$ และ $\int_{\text{all space}}\delta^3(\vec{r})\,dv = 1$ Position vector : $\vec{r} = x\vec{a}_x + y\vec{a}_y + z\vec{a}_z$ $\int_{\text{all space}} f(\vec{r})\,\delta^3(\vec{r}-\vec{a}) = f(\vec{a})$

ex Find $\nabla\cdot\left(\frac{1}{R^2}\vec{a}_R\right) = \nabla\cdot\left(\frac{\vec{r}}{r^3}\right)$

Sol^n $\nabla\cdot\left(\frac{1}{R^2}\vec{a}_R\right) = \frac{1}{R^2\sin\theta}\frac{\partial}{\partial R}\left(\frac{R^2\sin\theta}{R^2}\right) = 0$

ผลที่ได้นั้นน่าประหลาดใจ เพราะ จากรูปด้านซ้าย สนาม $\frac{1}{R^2}\vec{a}_R$ นั้นพุ่งออกจากจุดกำเนิดทุกทิศทาง

พิจารณา $\int_v \left(\nabla\cdot\frac{1}{R^2}\vec{a}_R\right) dv = \oint_S \left(\frac{1}{R^2}\vec{a}_R\right)\cdot d\vec{s}$

$$= \int_0^{2\pi}\int_0^{\pi}\frac{1}{R^2}\cdot R^2\sin\theta\, d\theta\, d\phi = 4\pi \quad \text{— (*)}$$

หากพิจารณาว่า $\nabla\cdot\left(\frac{1}{R^2}\vec{a}_R\right)$ เป็น 0 ทุกที่ยกเว้นที่จุดกำเนิด

จาก (*) และ $\int_{\text{all space}}\delta^3(\vec{r})\,dv = 1$ $\therefore$ $\boxed{\nabla\cdot\left(\frac{1}{R^2}\vec{a}_R\right) = 4\pi\delta^3(\vec{r})}$

Helmholtz's Theorem

A vector field (vector point function) is determined to within an additive constant if both its divergence and its curl are specified everywhere.

สนาม $\vec{F}$ ใดๆสามารถเขียนในรูป $\vec{F} = \underbrace{-\nabla V}_{\text{irrotational } (\nabla\times\vec{F}=0)} + \underbrace{\nabla\times\vec{A}}_{\text{solenoidal } (\nabla\cdot\vec{F}=0)}$

ex $\vec{F} = \vec{a}_x(3y - c_1 z) + \vec{a}_y(c_2 x - 2z) - \vec{a}_z(c_3 y + z)$.

(a) Determine the constant c_1, c_2, c_3 if $\vec{F}$ is irrotational.

(b) Determine the scalar potential function V. ($\vec{F} = -\nabla V$)

Sol^n (a) $\vec{F}$ is irrotational $\therefore \nabla\times\vec{F} = 0$

$$\nabla\times\vec{F} = \begin{vmatrix} \vec{a}_x & \vec{a}_y & \vec{a}_z \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 3y - c_1 z & c_2 x - 2z & -c_3 y - z \end{vmatrix} = \begin{bmatrix} -c_3 + 2 \\ -c_1 \\ c_2 - 3 \end{bmatrix}$$

$\therefore$ $\underline{c_1 = 0}$, $\underline{c_2 = 3}$, $\underline{c_3 = 2}$

(b) $\vec{F} = -\nabla V = -\vec{a}_x\frac{\partial V}{\partial x} - \vec{a}_y\frac{\partial V}{\partial y} - \vec{a}_z\frac{\partial V}{\partial z}$

$\frac{dV}{dx} = -3y$ — (1)

$\frac{dV}{dy} = -3x + 2z$ — (2)

$\frac{dV}{dz} = 2y + z$ — (3)

$\therefore$ $\underline{V = -3xy + 2yz + \frac{z^2}{2}}$

Additional Problems

① Let $\vec{\mathscr{R}}$ be the separation vector from a fixed point (x', y', z') to the point (x, y, z) and let $\mathscr{R}$ be its lenght. Show that

(a) $\nabla(\mathscr{R}^2) = 2\vec{\mathscr{R}}$

(b) $\nabla(1/\mathscr{R}) = -\vec{a}_{\mathscr{R}}/\mathscr{R}^2$

(c) What is the general formula for $\nabla(\mathscr{R}^n)$?

Sol^n $\vec{\mathscr{R}} = (x-x')\vec{a}_x + (y-y')\vec{a}_y + (z-z')\vec{a}_z$

(a) $\nabla(\mathscr{R}^2) = \nabla\left((x-x')^2 + (y-y')^2 + (z-z')^2\right)$

$$= 2(x-x')\vec{a}_x + 2(y-y')\vec{a}_y + 2(z-z')\vec{a}_z = 2\vec{\mathscr{R}}$$

(b) $\nabla(1/\mathscr{R}) = \nabla\left(\frac{1}{\sqrt{(x-x')^2 + (y-y')^2 + (z-z')^2}}\right)$

$$= -\frac{(x-x')\vec{a}_x + (y-y')\vec{a}_y + (z-z')\vec{a}_z}{\left((x-x')^2 + (y-y')^2 + (z-z')^2\right)^{3/2}}$$

$$= -\frac{\vec{\mathscr{R}}}{\mathscr{R}^3} = -\frac{\vec{a}_{\mathscr{R}}}{\mathscr{R}^2}$$

(c) $\nabla(\mathscr{R}^n) = \nabla\left\{\left[(x-x')^2 + (y-y')^2 + (z-z')^2\right]^{n/2}\right\}$

$$= n\left((x-x')^2 + (y-y')^2 + (z-z')^2\right)^{\frac{n-2}{2}} \times \left[(x-x')\vec{a}_x + (y-y')\vec{a}_y + (z-z')\vec{a}_z\right]$$

$$= n\mathscr{R}^{n-2}\vec{\mathscr{R}} = \underline{n\mathscr{R}^{n-1}\vec{a}_{\mathscr{R}}}$$

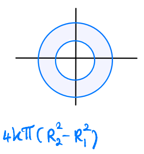
② Assuming that a cloud of electrons confined in a region between two spheres of radii 2 and 5 (cm) has a charge density of $\rho = \frac{-3\times10^{-8}}{R^4}\cos^2\phi$ (C/m^3) find the total charge contained in the region.

Sol^n $Q = \int_v \rho\, dv = \int_0^{2\pi}\int_0^{\pi}\int_{0.02}^{0.05}\left(-\frac{3\times10^{-8}}{R^4}\cos^2\phi\right)R^2\sin\theta\, dR\, d\theta\, d\phi$

$$= -3\times10^{-8}\cdot\int_0^{2\pi}\cos^2\phi\, d\phi\int_0^{\pi}\sin\theta\, d\theta\int_{0.02}^{0.05}\frac{1}{R^2}\,dR$$

$$= (-3\times10^{-8})(\pi)(2)(30) = \boxed{-1.8\pi\ \mu C.}$$

③ Given $\vec{F} = \vec{a}_R kR$, determine whether the divergence theorem holds for the shell region enclosed by spherical surfaces at $R = R_1$ and $R = R_2$ ($R_2 > R_1$) centered at the origin.



Sol^n $\nabla\cdot\vec{F} = \frac{1}{R^2\sin\theta}\left(\frac{\partial}{\partial R}(R^2\sin\theta\cdot kR)\right) = 3k$

$$\int_v(\nabla\cdot\vec{F})\,dv = \int_0^{2\pi}\int_0^{\pi}\int_{R_1}^{R_2} 3k\, R^2\sin\theta\, dR\, d\theta\, d\phi$$

$$= 3k\cdot\int_0^{2\pi}d\phi\int_0^{\pi}\sin\theta\, d\theta\int_{R_1}^{R_2}R^2\, dR = 4k\pi(R_2^3 - R_1^3)$$

$$\oint_S \vec{F}\cdot d\vec{s} = \int_0^{2\pi}\int_0^{\pi}(\vec{a}_R kR_2)\cdot\vec{a}_R R_2^2\sin\theta\, d\theta\, d\phi + \int_0^{2\pi}\int_0^{\pi}(\vec{a}_R kR_1)\cdot(-\vec{a}_R R_1^2\sin\theta\, d\theta\, dR)$$

$$= k(R_2^3 - R_1^3)\int_0^{2\pi}d\phi\int_0^{\pi}\sin\theta\, d\theta = 4k\pi(R_2^3 - R_1^3)$$

④ Find the divergence and curl of the function

$$\vec{A} = r(2+\sin^2\phi)\vec{a}_r + r\sin\phi\cos\phi\,\vec{a}_\phi + 3z\vec{a}_z$$

Sol^n $\nabla\cdot\vec{A} = \frac{1}{r}\cdot\frac{\partial}{\partial r}\left(r\cdot r(2+\sin^2\phi)\right) + \frac{1}{r}\frac{\partial}{\partial\phi}(r\sin\phi\cos\phi) + \frac{\partial}{\partial z}(3z)$

$$= 4 + 2\sin^2\phi + \cos^2\phi - \sin^2\phi + 3$$

$$= 7 + \sin^2\phi + \cos^2\phi = \underline{8}$$

$$\nabla\times\vec{A} = \frac{1}{r}\begin{vmatrix} \vec{a}_r & \vec{a}_\phi r & \vec{a}_z \\ \partial/\partial r & \partial/\partial\phi & \partial/\partial z \\ r(2+\sin^2\phi) & r\cdot r\sin\phi\cos\phi & 3z \end{vmatrix} = \underline{\vec{0}}$$

⑤ Show that $\vec{F} = yz\vec{a}_x + zx\vec{a}_y + xy\vec{a}_z$ can be written both as the gradient of a scalar and a curl of vector. Find the scalar and vector potentials for $\vec{F}$.

Sol^n $\nabla\cdot\vec{F} = \frac{\partial}{\partial x}(yz) + \frac{\partial}{\partial y}(zx) + \frac{\partial}{\partial z}(xy) = 0$ —(*)

$$\nabla\times\vec{F} = \begin{vmatrix} \vec{a}_x & \vec{a}_y & \vec{a}_z \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ yz & zx & xy \end{vmatrix} = 0 \quad \text{— (**)}$$

From (*), (**) : $\underline{\vec{F} = -\nabla V + \nabla\times\vec{A}}$

$\vec{F} = -\nabla V \Rightarrow \begin{cases} \frac{\partial V}{\partial x} = yz \\ \frac{\partial V}{\partial y} = zx \\ \frac{\partial V}{\partial z} = xy \end{cases}$

$\therefore$ $\underline{V = xyz}$

$\vec{F} = \nabla\times\vec{A} \Rightarrow \begin{cases} \frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z} = yz \\ \frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x} = zx \\ \frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} = xy \end{cases}$

(not unique)

$$\underline{\vec{A} = \frac{1}{2}\left[x(z^2-y^2)\vec{a}_x + y(x^2-z^2)\vec{a}_y + z(y^2-x^2)\vec{a}_z\right]}$$